A Kalman Filter for GLAST

• What is a Kalman Filter and how does it work.

• Overview of Implementation in GLAST

• Validation (or *Sea Trials*)

Reference: *Data Analysis Techniques in HEP* by Fruthwirth et al, 2000
The Kalman filter process is a successive approximation scheme to estimate parameters.

Simple Example: 2 parameters - intercept and slope: \( x = x_0 + S_x \times z; \quad P = (x_0, S_x) \)

Errors on parameters \( x_0 \) & \( S_x \): covariance matrix: \( C = \begin{pmatrix} C_{x-x} & C_{x-s} \\ C_{s-x} & C_{s-s} \end{pmatrix} \)

\( C_{x-x} = \langle (x-x_m)(x-x_m) \rangle \) \quad \text{In general} \quad C = \langle (P - P_m)(P - P_m)^T \rangle

Propagation:

\( x(k+1) = x(k) + S_x(k) \times (z(k+1) - z(k)) \)

\( P_{m(k+1)} = F(\delta z) \times P(k) \) where

\( F(\delta z) = \begin{pmatrix} 1 & z(k+1) - z(k) \\ 0 & 1 \end{pmatrix} \)

\( C_{m(k+1)} = F(\delta z) \times C(k) \times F(\delta z)^T + Q(k) \)
Kalman Filter (2)

Form the weighted average of the k+1 measurement and the propagated track model:

Weights given by inverse of Error Matrix: $C^{-1}$

Hit: $X(k+1)$ with errors $V(k+1)$

$$P(k+1) = \frac{Pm(k+1)Cm^{-1}(k+1) + X(k+1)V^{-1}(k+1)}{Cm^{-1}(k+1) + V^{-1}(k+1)}$$

and

$$C(k+1) = (Cm^{-1}(k+1) + V^{-1}(k+1))^{-1}$$

Now its repeated for the k+2 planes and so on. This is called FILTERING - each successive step incorporates the knowledge of previous steps as allowed for by the NOISE and the aggregate sum of the previous hits.
We start the FILTER process at the conversion point

BUT... We want the best estimate of the track parameters at the conversion point.

Must propagate the influence of all the subsequent Hits backwards to the beginning of the track - Essentially running the FILTER in reverse

This is call the SMOOTHER & the linear algebra is similar.

Residuals & $\chi^2$:

Residuals: $r(k) = X(k) - Pm(k)$

Covariance of $r(k)$: $Cr(k) = V(k) - C(k)$

Then: $\chi^2 = r(k)^T Cr(k)^{-1} r(k)$ for the $k^{th}$ step
Implementation in GLAST

- 3 Dimensional: Essentially GLAST is composed of 2 - 2D trackers however multiple scattering mixes x & y. This creates correlations between the two projections and hence the covariance matrix (C) has significant off (block) diagonal terms.
- Difference between two separate projections and 3D projection becomes increasingly important as BOTH the x & y become large.
- Calculation of $\chi^2$ involves both x & y and their correlation
- The SMOOTHed $\chi^2$ is not a true $\chi^2$ as errors are correlated point to point (not so for the FILTER $\chi^2$). However since the smallest errors (and hence the largest weights) are the measurement errors the difference between them is small. (Presently we use the SMOOTHed $\chi^2$)
$\chi^2$ and the 1-Event Display

3 Views of a 1 GeV $\mu^+$
End-to-End Testing

Objective: Test if the implementation of the errors in the Kalman Filter Routines is Correct.

Method: Use Monte Carlo μ’s (KE = 100 MeV, 1 GeV, & 10 GeV)
Provide the Kalman Filter with the correct energy (pβ)

Test: If Monte Carlo generation of multiple scattering is the same as that in the Kalman Filter AND the calculation of the covariance matrices is correct AND their usage is correct THEN we except $<\chi^2> \sim 1.0$ independent of position and angle

μ’s generated over $-1 < \cos(\theta) < 0$
The First Problem: \( \chi^2 > 1 \)

100 MeV - Normal Inc.

\( N_{\text{hits}} = 36 \)

\( \langle \chi^2 \rangle = 1.6 \)

Note: \( \mu \)'s generated with 100 MeV KE. This implies
\( E_{\text{tot}} = 205.7 \text{ MeV} \) and
\( p\beta = 151.4 \text{ MeV} \)

Red Line: \( \chi^2 \) function with parameters as above
Partial Solution - Include Energy Loss

100 MeV $\mu$'s entering the Tracker exit with $\sim 65$ MeV

First guess would give (assuming $\beta \sim 1$):
$\sim 22$ MeV ($50:50$ for Si+C:W)

Correcting for $\beta$ ($= .85$ const.):
$\sim 30$ MeV

Integrating over path
$\sim 35$ MeV

Implemented Bethe-Block Energy Loss in Kalman Filter (see results)

Problem becomes small by 1 GeV
Second Problem: $\chi^2$ Depends on Angles

1 GeV Muons

Dependence on $\cos(\theta)$
10 GeV Muons

All Angles

\( \theta < 45 \)

\( \theta > 45 \)
Cluster Size Error Dependence

Upper Plots: Error \sim (\text{Size} \times \sigma_p)

Resolution: Meas. Errors

Lower Plots: Error \sim \sigma_P

Where:

\[ \sigma_P = \frac{\sigma_P}{\sqrt{12}} \]

RED Line at \( \chi^2 = 1 \)
Third Problem: Tower Co-ordinate

**Issue:** To include or not include the Tower Co-ordinate as well as the Strip Co-ordinate.

Inclusion controlled by the mapping of the measurements onto the parameters and visa-versa. (usually called the $H$ matrix).

Reason not to include:
1) When results examined on a scale commensurate with bin size (Tower) binning effects appear.

2) Slight pull of fit toward center of tower at normal incidence.

3) Masks the $\chi^2$ behavior of the strip co-ordinate.
$\cos(\theta) = -1$

\begin{align*}
\langle N_{\text{hits}} \rangle &= 36 \\
\langle \chi^2 \rangle &= 1.25
\end{align*}

\begin{align*}
\langle N_{\text{hits}} \rangle &= 20 \\
\langle \chi^2 \rangle &= 1.4
\end{align*}

$-1 < \cos(\theta) < 0$

\begin{align*}
\langle \sigma_{\text{FIT}} \rangle &= 20.7 \text{ mrad} \\
\langle N_{\text{hits}} \rangle &= 20 \\
\langle \chi^2 \rangle &= 1.4 \text{ mrad}
\end{align*}
1 GeV $\mu^+$

$\cos(\theta) = -1$

$\langle N_{\text{hits}} \rangle = 36$

$\langle \chi^2 \rangle = 1.05$

$\langle \sigma_{\text{FIT}} \rangle = 3.4 \text{ mrad}$

$-1 < \cos(\theta) < 0$

$\langle N_{\text{hits}} \rangle = 22$

$\langle \chi^2 \rangle = 1.06$

$\langle \sigma_{\text{FIT}} \rangle = 4.0 \text{ mrad}$

Notice the Binning Effects?
$10 \text{ GeV } \mu^+$

$\cos(\theta) = -1$

$\langle N_{\text{hits}} \rangle = 36$
$\langle \chi^2 \rangle = 1.08$

$\langle \sigma_{\text{FIT}} \rangle = 0.61 \text{ mrad}$

$-1 < \cos(\theta) < 0$

$\langle N_{\text{hits}} \rangle = 24$
$\langle \chi^2 \rangle = 1.05$

$\langle \sigma_{\text{FIT}} \rangle = 0.63 \text{ mrad}$
Conclusions

A 3 dimensional Kalman Filter has been implemented

The errors, as reflected in $\chi^2$

- Are ~ not dependent on the Polar Angle ($\theta$)
- Are ~ not dependent on the Azimuthal Angle ($\phi$)
- **DO** dependent on energy:
  - remaining error in Kalman Multiple Scattering?
  - $G4$ give MS 20% large then Wallet Card Formulas?

The match of $\chi^2$ distributions to the ideal case is reasonable.
Energy Dep. Suspicion

1) The match of $\chi^2$ to $\chi^2$ functions is good in shape for leading edge

2) At high energy the match is good overall

Its as if the usual Multiple Scattering in G4 is as expected, but occasionally there is a BIG scatter which skews $\chi^2$.

You can see this in the One Event Display!

Is this Physical?
Could δ-rays be the source?

Maybe - but for sure examples as shown at the left cause large contributions to $\chi^2$

SSD viewed edge on

δ-ray
Strip & Cluster Meas. Errors

Track Measurement Error $\delta_{\text{meas}} = (W_{\text{cls}} - W_{\text{proj}})/\sqrt{12}$

Predicted effects (wish-list):

1) Unweight oversized clusters on track - lessening effect of $\delta_{\text{rays}}$

2) Tighten hit locations in cases where $W_{\text{cls}} \sim W_{\text{proj}}$

This could result in improved angular resolution at high energy